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On the Unrestricted Iteration of Projections in Hilbert Space

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Beginning in 1869 with the alternating method of Schwarz [22] for solving the Dirichlet problem for Laplace's equation, iterations of projections and their convergence properties have played an important role in mathematics. Perhaps they were first recognized for their intrinsic interest by J. von Neumann in 1933, through his well-known alternating projection theorem [18]. It asserts that if P and Q are orthogonal projections onto closed subspaces S_P and S_Q of a Hilbert space H, then the sequence of products P, QP, PQP, ..., converges strongly to the projection onto the subspace $S_P \cap S_Q$. See also Wiener [28].

With some modifications, von Neumann's theorem and its extensions find application in domain decomposition methods for the numerical solution of partial differential equations [3, 16, 27], linear inequalities [24], approximation theory [20, 11], population biology [12], mathematical programming [6, 2, 15], and in the broad field of image recovery [25, 29] (most notably, computer tomography [23, 14, 19]).

In light of the broad utility of von Neumann's original result, it is surprising that the natural generalization has gone unresolved, namely,

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(*) if P, Q, and R are orthogonal projections onto respective closed subspaces S_P, S_Q, and S_R of a Hilbert space, does an unrestricted iteration converge strongly to the projection onto the subspace $S_P \cap S_Q \cap S_R$?

In 1965, Amemiya and Ando [1] proved weak convergence, for nonexpansive maps merely satisfying the following condition (W), namely, if whenever the sequence $\{v_n\}$ is bounded and $|v_n| - |Tv_n| \to 0$, it follows that the weak $\lim_{n \to \infty} (v_n - Tv_n) = 0$. (In fact, their result continues to hold in a smooth reflexive Banach space [10].) Powers of a *single* (W) map may fail to converge strongly [9]. However, if the map satisfies the stronger condition (S), which is similar to (W) except that weak is replaced by strong convergence, then its powers do converge strongly.

In 1962, I. Halperin [13] introduced condition (K) with constant $k > 0$ for a (nonexpansive) map T, namely, that $|x - Tx|^2 \leqslant k(|x|^2 - |Tx|^2)$ for all $x \in H$. It is easy to see that condition (K) is at least as strong as (S), yet we show that an (unrestricted) iteration of two maps, each satisfying (K), can fail to converge strongly. One may conclude that if (*) is true, it is due at least in part to the particular geometric properties of projections. Indeed, it is this aspect of projections that we shall emphasize.

In Example 1 we show that condition (S) and condition (K) are not equivalent. Example 2, immediately following, contains the aforementioned counterexample.

We say that the (algebraic) *semigroup* $\mathrm{S}(Q_1, Q_2, ..., Q_N)$ generated by the N projections $\{Q_j : 1 \leqslant j \leqslant N\}$ has condition (K) with constant k if for any $x \in H$, and each word $W \in \mathrm{S}$, $|x - Wx|^2 \leqslant k(|x|^2 - |Wx|^2)$. (For descriptive purposes, we shall henceforth write $W = S_n = Q_{r(n)} Q_{r(n-1)} \cdots Q_{r(1)}$, where r is a self-mapping of the set of natural numbers, and refer to S_n as a random or unrestricted product of the Q_j's.) In the presence of this condition, the sequence $(S_n x)$, for $x \in H$, is easily seen to be Cauchy and hence convergent. We are indebted to R. E. Bruck for suggesting use of (K) in this context.

In Proposition 1 we provide a geometric proof of the fact that the semigroup generated by two projections has condition (K) with $k = 2$. Example 3 shows that this constant cannot be improved.

Example 4 shows that the three projection case is fundamentally more complicated than the two projection case. Nevertheless, our main theorem states that the semigroup generated by the orthogonal projections onto N one-dimensional subspaces of a Hilbert space has condition (K) with constant $k = N$. We feel that this result can contribute towards the resolution of (*).

We conclude with a property of a related semigroup (Proposition 2).

For other recent results on iterations of (linear and nonlinear) nonexpansive maps see, for example, [5, 4, 21, 8, 9].

EXAMPLE 1. Let H be a (real) separable Hilbert space with orthonormal basis (e_j) for $j=0, 1, 2, \ldots$. One may define a bounded linear operator T on H such that

$$Te_j = \cos\frac{1}{j}\, e_j + \left(\sin\frac{1}{j} - \frac{1}{j^2}\right) e_{j+1} \qquad \text{for } j \text{ odd},$$

and

$$Te_j = 0 \qquad \text{for } j \text{ even}$$

Note that $\|T\| \leqslant 1$ since

$$\cos^2\frac{1}{j} + \left(\sin\frac{1}{j} - \frac{1}{j^2}\right)^2 < 1 \qquad \text{for} \quad j = 1, 2, \ldots.$$

Writing $x_n = \sum_j a_j^{(n)} e_j$, we have

$$|Tx_n|^2 = \sum_{j\ \mathrm{odd}} (a_j^{(n)})^2 \left(\cos^2\frac{1}{j} + \left(\sin\frac{1}{j} - \frac{1}{j^2}\right)^2\right).$$

Now if $|x_n| \to 1$ and $|Tx_n| \to 1$ then one may easily see that $\sum_{j\ \mathrm{even}} (a_j^{(n)})^2 \to 0$ and for fixed odd j, $a_j^{(n)} \to 0$.

Since both series

$$\sum \left(\sin\frac{1}{j} - \frac{1}{j^2}\right)^2 \qquad \text{and} \qquad \sum \left(1 - \cos\frac{1}{j}\right)^2$$

are convergent, an easy argument gives that $|(1-T)\, x_n|^2 \to 0$, i.e., T has condition (S).

For T to have condition (K), the quotient

$$\frac{|(1-T)\, e_j|^2}{1 - |Te_j|^2}$$

must be bounded for all $j = 1, 2, \ldots$. Equivalently,

$$\frac{(1-\cos u)^2 + (\sin u - u^2)^2}{1 - \cos^2 u - (\sin u - u^2)^2}$$

must remain bounded as $u \downarrow 0$. However, two applications of L'Hospital's rule reveal that

$$\lim_{u \downarrow 0} \frac{(1-\cos u)^2 + (\sin u - u^2)^2}{1 - \cos^2 u - (\sin u - u^2)^2} = +\infty.$$

We conclude that T cannot have condition (K).

EXAMPLE 2. Let H be as in Example 1. Given $k_1 > 0$ and $k_2 > 0$, we shall exhibit two maps, each satisfying condition (K) with k_1 and k_2, respectively, and construct a random product from them that fails to converge strongly.

Define a bounded linear operator T on the two dimensional subspace $\langle e_i, e_{i+1} \rangle$ by the matrix

$$b_i \begin{bmatrix} \cos \alpha_i & \sin \alpha_i \\ -\sin \alpha_i & \cos \alpha_i \end{bmatrix}.$$

We claim that given any fixed $k > 0$, there exist constants α_i and b_i such that

(1) T has (K) with constant k (and no smaller) and

(2) Given $\varepsilon > 0$ there exists an integer n_i such that $T^{n_i} e_i = \rho e_{i+1}$, where $\rho \geqslant 1 - \varepsilon$.

To prove (1) we first note that since T is a rotation multiplied by a constant, it suffices to verify that

$$|e_i - Te_i|^2 = k(|e_i|^2 - |Te_i|^2).$$

Let $l = |e_i - Te_i|$ and $b = |Te_i|$. So by the law of cosines we require that

$$l^2 = k(1 - b^2) = 1 + b^2 - 2b \cos \alpha_i.$$

If we pick $\alpha_i > 0$ small enough so that $\cos^2 \alpha_i \geqslant 1 - k^2$, then we can solve for b_i, obtaining

$$b_i = \frac{\cos \alpha_i + \sqrt{k^2 - 1 + \cos^2 \alpha_i}}{1 + k}.$$

This choice of b_i satisfies (1). An application of L'Hospital's rule shows that $(b_i)^{1/\alpha_i} \to 1$ as $\alpha \downarrow 0$. Hence there exists an integer n_i so that if $\alpha_i = \pi/2n_i$ then

$$T^{n_i} e_i = \rho e_{i+1} \qquad \text{with} \quad \rho \geqslant 1 - \varepsilon.$$

This fulfills (2), and the claim is established.

Now for $k_1 > 0$. We define T_1 on H as follows. Breaking H down into orthogonal subspaces $M_{2i} = \langle e_{2i}, e_{2i+1} \rangle$ for $i = 0, 1, 2, \ldots$, we let the restriction of T_1 to M_{2i} be a map of type T as above. That is, we may assume T_1 on M_{2i} to have condition (K) with constant k_1, and angle α_i and b_i such that for some integer n_{2i},

$$T_1^{n_{2i}} e_{2i} = \rho_{2i} e_{2i+1}, \qquad \text{with} \quad \rho_{2i} \geqslant 1 - \frac{1}{2^{2i+1}}.$$

Since the subspaces M_{2i} are reducing for T_1, we have that T_1 satisfies condition (K) with constant k_1 on H.

We next define a map T_2, which is similar to T_1 except it satisfies condition (K) with constant k_2, has reducing subspaces $\langle e_{2i+1}, e_{2i+2}\rangle$ for $i = 0, 1, 2, \ldots$ and satisfies

$$T_2^{n_{2i+1}} e_{2i+1} = \rho_{2i+1} e_{2i+2} \qquad \text{with} \quad \rho_{2i+1} \geqslant 1 - \frac{1}{2^{2(i+1)}}.$$

Define a random product S_n of these two maps as follows. We let

$$S_{n_0} e_1 = T_1^{n_0} e_1, \qquad S_{n_0+n_1} e_1 = T_2^{n_1} T_1^{n_0} e_1, \qquad S_{n_0+n_1+n_2} e_1 = T_1^{n_2} T_2^{n_1} T_1^{n_0} e_1,$$

etc. Since $F(T_1) = F(T_2) = (0)$, we know by [1] that S_n converges weakly to zero. Hence

$$|S_{n_0+n_1+\cdots+n_j} e_1| \geqslant \prod_{i=0}^{j} \left(1 - \frac{1}{2^{i+1}}\right)$$

must diverge to zero as $n \to \infty$, if S_n converges strongly. This is clearly not the case, as $\sum (1/2^i)$ converges.

We remark in passing that since

$$\frac{|e_{2i} - T_1^{n_{2i}} e_{2i}|^2}{1 - \rho_{2i}^2} \to \infty,$$

we observe that condition (K) with constant $k > 0$ for a map does not imply that its powers have condition (K) with that k.

We now note that in von Neumann's setting, $\mathbf{S}(P, Q)$ has condition (K). This was proved in [8]. However, we give an alternative proof, whose virtue lies in its entirely elementary (geometric) character.

PROPOSITION 1. *Let P and Q be orthogonal projection operators on a Hilbert space H. Then the algebraic semigroup $\mathbf{S}(P, Q)$ has* (K) *with constant $k = 2$.*

Proof. We may assume without loss of generality that $S_1 = P$, $S_2 = QP$, $S_3 = PQP, \ldots$. Let x_1 be any vector such that $Px_1 = x_1$, and let $x_2 = Qx_1$, $x_3 = Px_2$, $x_4 = Qx_3, \ldots$. We begin by establishing

$$|x_i - x_{i+n}|^2 \leqslant |x_i|^2 - |x_{i+n}|^2, \tag{3}$$

for all i and $n \geqslant 1$.

Assume $n = 1$. Then $|x_i|^2 = |x_i - x_{i+1}|^2 + |x_{i+1}|^2$ by the Pythagorean Theorem. Proceeding inductively, assume (3) has been established for $n > 1$ and all i. We will prove that (3) holds for all i and for $n + 1$.

Assume i is odd and n is even. Again by Pythagoras,

$$|x_i - x_{i+n+1}|^2 = |x_i - x_{i+1}|^2 + |x_{i+1} - x_{i+n+1}|^2. \tag{4}$$

Applying (3) with $n=1$ and the inductive assumption we obtain $|x_i - x_{i+n+1}|^2 \leqslant |x_i|^2 - |x_{i+1}|^2 + |x_{i+1}|^2 - |x_{i+n+1}|^2 = |x_i|^2 - |x_{i+n+1}|^2$. Thus (3) holds for i odd and $n+1$, n even.

Next, assume i is odd and n is odd. Appropriate application of the Pythagorean Theorem gives

$$|x_i - x_{i+n+1}|^2 = |x_i - x_{i+n}|^2 - |x_{i+n} - x_{i+n+1}|^2. \tag{5}$$

Again applying (3) with $n=1$ and the inductive assumption gives

$$|x_i - x_{i+n+1}|^2 \leqslant |x_i|^2 - 2\,|x_{i+n}|^2 + |x_{i+n+1}|^2.$$

Since $|x_{i+n+1}| \leqslant |x_{i+n}|$, the expression just obtained is less than or equal to $|x_i|^2 - |x_{i+n+1}|^2$. This proves (3) for $n+1$, n odd. Thus (3) holds for i odd and for $n+1$. An entirely similar argument yields the same conclusion for $n+1$ and i even. This completes the induction and proves the assertion (3).

To complete the proof, take an arbitrary vector x and positive integer n. Then $|x - x_n|^2 \leqslant 2\,|x - x_1|^2 + 2\,|x_1 - x_n|^2 \leqslant 2(|x|^2 - |x_1|^2 + |x_1|^2 - |x_n|^2) = 2(|x|^2 - |x_n|^2)$, by (3).

Proposition 1 still leaves open the question whether its conclusion holds for some $k < 2$. The following example settles this.

EXAMPLE 3. We consider 2×2 matrices

$$Q = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad P = \begin{bmatrix} \lambda & \sqrt{\lambda(1-\lambda)} \\ \sqrt{\lambda(1-\lambda)} & 1-\lambda \end{bmatrix},$$

where $0 < \lambda < 1$. It is evident that $P = P_\lambda$ and Q are projections (self-adjoint idempotents). Put $S = QP$.

In [8], it is proven that the following conditions on a contraction T are equivalent:

(i) T satisfies condition (K) with constant $k > 0$, and

(ii) $(I + T^*T)/2 \leqslant k/(1+k) + \operatorname{Re} T/(1+k)$.

We wish to find the largest $t = 1/(1+k)$ in $[0, 1]$ for which

$$\frac{I + S^*S}{2} \leqslant 1 - t + t \operatorname{Re} S, \qquad \text{for all } 0 < \lambda < 1.$$

In matrix terms, we want the largest t in the interval $[0, 1]$ such that, with respect to the usual operator order on operators on two-dimensional complex Hilbert space,

$$0 \leqslant \begin{bmatrix} 1-2t+2\lambda t-\lambda^2 & (t-\lambda)\sqrt{\lambda(1-\lambda)} \\ (t-\lambda)\sqrt{\lambda(1-\lambda)} & 1-2t-\lambda(1-\lambda) \end{bmatrix}.$$

The requirement that this matrix be $\geqslant 0$ imposes three conditions on the matrix in question, namely, that each of the diagonal terms be $\geqslant 0$ and that the determinant be $\geqslant 0$.

Let $A = 1-2t+2\lambda t-\lambda^2 = (1-\lambda)(\lambda-2t+1)$ and $B = 1-2t-\lambda(1-\lambda)$ be the diagonal terms. Then the determinant is $AB-(t-\lambda)^2\,\lambda(1-\lambda)$. Dividing out the $(1-\lambda)$ term and simplifying, we see that the condition for the determinant to be $\geqslant 0$ reduces to the inequality

$$\sqrt{\lambda} \leqslant \frac{1-2t}{t}. \tag{6}$$

We want (6) to hold for all λ, $0<\lambda<1$. Thus we want

$$1 \leqslant \frac{1-2t}{t}, \qquad \text{or} \qquad t=\frac{1}{3}, \qquad \text{or} \qquad k+1=3.$$

When $t=\frac{1}{3}$, one can check easily by calculus that the diagonal terms A and B are $\geqslant 0$ for all $\lambda \in (0, 1)$. In terms of Proposition 1, it is now evident that $k=2$ is the best possible value of k.

The special simplicity of the two projection case we have been discussing is best seen in light of the theory of von Neumann algebras. See [26]. J. Dixmier has shown [7] that the von Neumann algebra $\mathbf{M}$ generated by two projections P and Q contains a central projection Z such that $\mathbf{M}_Z$ is abelian and $\mathbf{M}_{I-Z}$ is of type I_2, that is, can be represented as 2×2 matrices over an abelian von Neumann algebra. Assume $Z=0$. Then P and Q can be represented by matrices

$$Q=\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \qquad \text{and} \qquad P=\begin{bmatrix} T & \sqrt{T(1-T)} \\ \sqrt{T(1-T)} & 1-T \end{bmatrix}, \qquad 0 \leqslant T \leqslant I.$$

(See [26] for a modern treatment.) The procedure in Example 3 was based simply on choosing various possible spectral values of T. In striking contrast to this result are phenomena in the three projection case which suggest that entirely new ideas will be needed to extend Proposition 1 to the case of $N \geqslant 3$ projections:

EXAMPLE 4. Let H be an infinite dimensional separable Hilbert space. We assert there exist three projections P, Q, and R, two of which commute with each other, such that the von Neumann algebra generated by P, Q, and R is a factor of type II_1. We are indebted to M. Takesaki for suggesting study of the particular example we use.

First, let $G = Z_2$, that is, the additive group of integers mod 2. For convenience, we use multiplicative notation and write $G = \{e, x\}$, with $x^2 = e$. A simple analysis reveals four projections in the group algebra of G: 0, e, $p = \frac{1}{2}e + \frac{1}{2}x$, and $\frac{1}{2}e - \frac{1}{2}x$. Note that any subalgebra of the group algebra containing p and e also contains $x = 2p - e$.

Next, let $G = Z_3$, the additive group of integers mod 3, which again we write in multiplicative notation as $\{e, y, y^2\}$, with $y^3 = e$. A straightforward but lengthy computation exhibits eight projections in the group algebra of G, namely, 0, e, $\frac{2}{3}e - \frac{1}{3}y - \frac{1}{3}y^2$, $s = \frac{1}{3}e + \frac{1}{3}y + \frac{1}{3}y^2$, $q = \frac{2}{3}e + (\omega/3)\, y + (\bar{\omega}/3)\, y^2$, $r = \frac{2}{3}e + (\bar{\omega}/3)\, y + (\omega/3)\, y^2$, $\frac{1}{3}e + (\omega^2/3)\, y + (\bar{\omega}^2/3)\, y^2$, and $\frac{1}{3}e + (\bar{\omega}^2/3)\, y + (\omega^2/3)\, y^2$, where $\omega = \frac{1}{2} + i(\sqrt{3}/2)$. Note that the linear span of e, q, and r contains $3(q - \frac{2}{3}e) + 3(r - \frac{2}{3}e) = y + y^2$ and $(3/i\sqrt{3})(q - \frac{2}{3}e) - (3/i\sqrt{3})(r - \frac{2}{3}e) = y - y^2$. Thus the linear span in question contains y and y^2. Also, $rq = qr = s$. Both group algebras in these examples are abelian.

Let $G = PSL(2, Z)$ denote the modular group. In G we distinguish elements S and T defined by

$$S = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \quad \text{and} \quad T = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

We note $S^2 = I$, $(ST)^3 = I$, and G is generated by S and T. In fact $PSL(2, Z) \cong Z_2 * Z_3$. Now, it is also known that for each element W of G, $W \neq$ identify, the conjugate class ZWZ^{-1} $(Z \in G)$ is infinite. This statement is abbreviated by saying that G is an *ICC* group.

We employ a construction first devised by Murray and von Neumann [17]. (See also [26, Proposition 7.9].) Let G be a countably infinite group with the *ICC* property. Let $H = l_2(G)$. Associate with each $g \in G$ the (unitary) operator $u(g)$ on H defined by $(u(g)\, f)(h) = f(g^{-1}h)$ $(f \in H)$ and let $\mathbf{M}$ be the von Neumann algebra generated by these operators $u(g)$ as g ranges over G. Then, $\mathbf{M}$ is a factor of type II_1.

For the group $G = PSL(2, Z)$, let $\mathbf{M}$ be its left ring. As G is *ICC*, $\mathbf{M}$ is a II_1 factor. By the identification of the generator x of Z_2 with S, we see that $P = \frac{1}{2}I + \frac{1}{2}S$ is a projection in $\mathbf{M}$ and $S = 2P - I$. Similarly, by the identification of the generator y of Z_3 with ST, we see that $Q = \frac{2}{3}I + (\omega/3)\, ST + (\bar{\omega}/3)(ST)^2$ and $R = \frac{2}{3}I + (\bar{\omega}/3)\, ST + (\omega/3)(ST)^2$ are commuting projections in $\mathbf{M}$ whose linear span with I contains ST and $(ST)^2$. Thus, the von Neumann subalgebra of $\mathbf{M}$ generated by P, Q, and R contains all operators

$u(g)$ $(g \in G)$ and therefore coincides with **M**. This affirms the assertion made at the beginning of the example, namely, that on a separable infinite dimensional Hilbert space, there exist three projection P, Q, and R, such that $QR = RQ$ and the von Neumann algebra generated by P, Q, and R is a factor of type II_1.

We now prove that in the special case of N projections onto lines (through the origin) in a Hilbert space, the (algebraic) semigroup **S** has condition (K) with $k = N$. That a bound k exists was predicted in [8]. However, the value of our result lies both in the suggestive number N (which may be useful in determining rates of convergence), and in knowing for which geometric configurations the bound is approached. Such information may be crucial in extending the following theorem to more general projections.

THEOREM. *Let Q_i, $1 \leqslant i \leqslant N$, be the orthogonal projections onto N one-dimensional subspaces of a (real) Hilbert space. Then* $\mathbf{S}(Q_1, Q_2, ..., Q_N)$ *has condition* (K) *with constant $k = N$.*

The proof consists of the following observations and lemmata.

First we observe that the theorem is true if and only if

$$\varphi_w = \frac{|e_1 - We_1|^2}{1 - |We_1|^2} \leqslant N$$

for all semigroups $\mathbf{S}'(Q'_1, Q'_2, ..., Q'_N)$, where the Q'_i's are orthogonal projections (onto lines through the origin), $W \in \mathbf{S}'$, and $We_1 \neq e_1$. We emphasize that N is assumed to be fixed throughout the proof.

We call the sequence of points $(e_1, S_1 e_1, S_2 e_1, ..., S_m e_1)$ the *nodes for the projection path* for the product S_m. We assume without loss of generality that $S_1 e_1 \neq e_1$, $S_m e_1 \neq \alpha e_1$ for $0 \leqslant \alpha < 1$ and $N \geqslant 2$ for the remainder of the proof.

When the final projection $Q = Q_{r(m)}$ of an otherwise variable word W is fixed, it will be convenient to designate $l = |e_1 - We_1|$ and $b = |We_1|$, so that

$$\varphi_W = \frac{l^2}{1 - b^2}.$$

We will refer to b as the *ending* norm for the path determined by the word W. The point $S_m e_1$ defines a ray r_m and we will denote by θ the angle (of radian measure $\leqslant \pi$) between e_1 and this r_m. By trigonometry we see that

$$\varphi = 1 + \frac{2b(b - \cos\theta)}{1 - b^2},$$

and that φ is an increasing function of b.

LEMMA 1. *Given any path P, there exists a path P_1 with at most two nodes on each line, (at most one on each side of the origin), such that* $\varphi_P \leqslant \varphi_{P_1}$.

Proof. Suppose a line l_k is visited twice on the same side of the origin in the product S_m. Explicitly, we have

$$S_m e_1 = S_{r(m)} \cdots S_{r(j)} Q_k \cdots Q_k S_{r(i)} \cdots e_1.$$

Because

$$Q_k S_{r(i)} \cdots e_1 = \alpha Q_k \cdots Q_k S_{r(i)} \cdots e_1$$

for some $\alpha \geqslant 1$, we have, for

$$S'_m e_1 = S_{r(m)} \cdots S_{r(j)} Q_k S_{r(i)} \cdots e_1$$

that $|S_m e_1| \leqslant |S'_m e_1|$. By the observation preceding the lemma we conclude that $\varphi_{S_m e_1} \leqslant \varphi_{S'_m e_1}$.

LEMMA 2. *Given any path* $P = (e_1, S_1 e_1, S_2 e_1, ..., S_m e_1)$, *there exists a planar path P_2 of $m+1$ nodes contained entirely in a half of a plane containing e_1, $S_m e_1$, and the origin, and ending on the ray determined by $S_m e_1$, such that $b_P \leqslant b_{P_2}$, and therefore,* $\varphi_P \leqslant \varphi_{P_2}$.

(We note for the sake of clarity that N lines give rise to possibly $2N$ rays, so that the final planarized path, though restricted to a half plane, may appear to involve more lines than the original path.)

Proof. We assume e_1, $S_m e_1$, and the origin are not collinear; if they are, we proceed with the following argument, with $S_{m-1} e_1$ in place of $S_m e_1$. Set $v = S_m e_1$. Let M denote the plane generated by v, e_1, and the origin and having orthogonal unit vectors e_1 and e_2 and set $z = S_{m-1} e_1$. Let R_{m-1} be a rotation about e_1 aligning z with M so that v, z, and e_1 all share the same half plane. Applying R_{m-1} to all the preceding nodes, we obtain a new projection path on $m-1$ new (rotated) rays, which we denote S'_{m-1}. Clearly

$$|z| = |S'_{m-1} e_1|. \tag{7}$$

If we set $z' = S'_{m-1} e_1$ our claim is that

$$|Q_{r(m)} z| \leqslant |Q_{(m)} z'|. \tag{8}$$

To prove the claim, we set

$$v = a_1 e_1 + a_2 e_2$$

$$z = b_1 e_1 + b_2 e_2 + y$$

$$z' = b_1 e_1 + b'_2 e_2,$$

where $y \in M^{\perp}$ and the e_1 components of z and z' are the same by (7) and the fact that R_{m-1} is a rotation about e_1. The relation (8) will follow if the angle between v and z' is larger than the the angle between v and z. Equivalently, in this setting, we must verify that

$$(v, z) = a_1 b_1 + a_2 b_2 \leqslant a_1 b_1 + a_2 b_2' = (v, z'). \tag{9}$$

From the initial geometry (before the rotation) and the choice of rotation, one sees that a_2, b_2, and b_2' all have the same sign. By (7) we have $b_1^2 + b_2^2 + y^2 = b_1^2 + b_2'^2$ implying (9).

We now continue the process, sequentially aligning all the prior nodes (and rays) into M. Each operation yields a new system of rays whose final ending norm b on the ray determined by v is higher than that of the old configuration. If we had made the substitution described in the first sentence of the proof, we would have applied $Q_{r(m)}$ to $v = S_{m-1} e_1$ and, by a simple argument as in Lemma 1, realized a higher ending norm b on the ray determined by $S_m e_1$. This completes the proof.

Suppose the path $P = (e_1, S_1 e_1, S_2 e_1, ..., S_m e_1)$ is both of type P_1 and P_2, i.e., it is contained in a half plane containing e_1 and $S_m e_1$ and each ray is visited once. Let r_i, $1 \leqslant i \leqslant m$, denote the ray determined by $S_1 e_1$, and θ_i the angles between the corresponding rays and e_1. If the angles θ_i do not form a strictly increasing sequence, we say that the path backs up.

LEMMA 3. *Suppose the path P is both of type P_1 and P_2 and has a backup. Then there exists a backup free path P_3 determined by a proper subset of the rays of P (involving no more than m nodes), starting at e_1 and ending on the ray determined by $S_m e_1$, which has a higher ending norm b than P.*

Proof. Consider first that portion of the path on rays r_i with angle less than θ_m. Note the first ray r_i for which the very next angle θ_{i+1} is smaller than θ_i. We look to the next occurrence in P of an angle θ_j such that $\theta_j > \theta_i$. We observe that by deleting the rays between r_{i+1} and r_j, we obtain a path P' (of fewer nodes) which jumps directly from r_i to r_j, and at ray r_j, has a higher norm than P does on the same ray. Hence P' also has a higher *ending* norm b on r_m than P. We continue this process, if necessary, eliminating all backups involving rays of angle less than θ_m. If no further backups occur in the path P', then we use P' for P_3, proving the claim.

If angles greater than θ_m occur, we note the last ray r_k in P' before such an angle. By deleting all the intervening rays between r_k and r_m, we create a path which we denote by P'', which ends on r_m and has a higher ending norm b than P', and which has no backups. We use P'' for P_3, finishing the argument.

We mention in passing that if $P=(e_1, S_1e_1, S_2e_1, ..., S_me_1)$ is a path of type P_3, one may insert *additional* rays between e_1 and r_m and produce the obvious path P^* without backups. We note that P^* has a higher ending norm on r_m than P. This fact will be used implicitly further on.

LEMMA 4. *For a planar path P of type P_2 there exists an "equi-angular" path P_4, ending on the ray determined by S_me_1 such that $b_{P_2} \leqslant b_{P_4}$.*

Proof. We may assume that P is of type P_3, by Lemma 3.

From geometrical considerations we see that a maximum ending norm b will occur for some configuration of $m-1$ rays between e_1 and S_me_1. Denote the intervening angles by $\alpha_i, 0 \leqslant \alpha_i \leqslant \pi/2, 1 < i < m$. Let θ be the (fixed) angle between e_1 and S_me_1. So we seek to maximize

$$b=\cos\alpha_1 \cos\alpha_2 \cdots \cos\alpha_m$$

subject to

$$\alpha_1+\alpha_2+\cdots+\alpha_m-\theta=0.$$

Using the method of Lagrange multipliers, we obtain the following system of equations:

$$\begin{aligned} -\sin\alpha_1 \cos\alpha_2 \cdots \cos\alpha_m &= \lambda \\ -\cos\alpha_1 \sin\alpha_2 \cdots \cos\alpha_m &= \lambda \\ \vdots \quad \vdots \qquad \vdots \quad & \\ -\cos\alpha_1 \cos\alpha_2 \cdots \sin\alpha_m &= \lambda. \end{aligned}$$

Cross multiplication now gives $\tan\alpha_1 = \tan\alpha_2 = \cdots = \tan\alpha_m$.
The constraints on the α's give $\alpha_1=\alpha_2=\cdots=\alpha_m$.

Some trigonometric inequalities will be necessary for the remainder of the proof.

LEMMA 5. *For $n \geqslant 3$ and $0<\alpha \leqslant \pi/n$ we have*

$$1-\frac{n}{2}\alpha^2 < \cos^n\alpha < 1-\frac{2n}{\pi^2}\alpha^2.$$

Proof. To prove the right hand side of the inequality, we have to show that

$$\begin{aligned} \frac{2n}{\pi^2}\alpha^2 < 1-\cos^n\alpha &= (1-\cos\alpha)(1+\cos\alpha+\cdots+\cos^{n-1}\alpha) \\ &= \left(2\sin^2\frac{\alpha}{2}\right)(1+\cos\alpha+\cdots+\cos^{n-1}\alpha), \end{aligned}$$

or, equivalently,

$$\frac{\sin^2 \dfrac{\alpha}{2}}{\left(\dfrac{\alpha}{2}\right)^2}(1+\cos\alpha+\cdots+\cos^{n-1}\alpha)>\frac{4n}{\pi^2}.$$

Now, $\sin x/x$ is decreasing for $0<x<\pi/2$, so that

$$x\leqslant\frac{\pi}{3}\Rightarrow\left(\frac{\sin(\alpha/2)}{(\alpha/2)}\right)^2\geqslant\left(\frac{\sin(\pi/6)}{(\pi/6)}\right)^2=\left(\frac{3}{\pi}\right)^2.$$

Also,

$$\cos\alpha>1-\frac{\alpha^2}{2}\qquad\text{for}\quad 0<x<\pi/2,$$

and

$$\left(1-\frac{\alpha^2}{2}\right)^n>1-\frac{n\alpha^2}{2}\qquad\text{for}\quad 0<\alpha<\sqrt{2}.$$

Hence

$$\begin{aligned}1+\cos\alpha+\cdots+\cos^{n-1}\alpha&>1+\left(1-\frac{\alpha^2}{2}\right)+\left(1-\frac{2\alpha^2}{2}\right)\\+\cdots+\left(1-\frac{n-1}{2}\alpha^2\right)&=n-\frac{\alpha^2}{2}(1+2+\cdots+n-1)\\&=n-\frac{\alpha^2}{2}\frac{(n-1)n}{2}=n\left(1-\frac{(n-1)\alpha^2}{4}\right)\\&>n\left(1-\frac{(n-1)\pi^2}{4n^2}\right).\end{aligned}$$

Thus it suffices to show that

$$\left(\frac{9}{\pi^2}\right)n\left(1-\frac{(n-1)\pi^2}{4n^2}\right)>\frac{4n}{\pi^2}.$$

Equivalently,

$$\frac{(n-1)\pi^2}{4n^2}<\frac{5}{9},$$

or

$$\frac{n-1}{n^2}<\frac{20}{9\pi^2}\approx 0.2251.$$

This is true for $n\geqslant 3$.

The other side of the inequality follows from

$$(\cos\alpha)^n > \left(1-\frac{\alpha^2}{2}\right)^n > 1-\frac{n\alpha^2}{2}.$$

LEMMA 6. *For $n \geqslant 2$ and $0<\alpha\leqslant \pi/n$ we have*

$$\cos^n \alpha > \frac{\sin n\alpha - \sin\alpha}{\sin(n-1)\,\alpha}.$$

Proof. We note that

$$0<\alpha<\frac{\pi}{n-1} \Rightarrow \frac{(n-1)\,\alpha}{2}<\frac{\pi}{2} \qquad \text{for} \quad n \geqslant 2.$$

Now

$$\frac{\sin n\alpha-\sin\alpha}{\sin(n-1)\,\alpha}=\frac{\cos((n+1)\,\alpha/2)}{\cos((n-1)\,\alpha/2)},$$

and we wish to show that this quantity is less than $\cos^n\alpha$ for $0<\alpha<\pi/n$. So, we wish to show that

$$\cos\frac{(n+1)\,\alpha}{2}<\cos^n\alpha\cos\frac{(n-1)\,\alpha}{2} \qquad \text{for} \quad 0<\alpha<\frac{\pi}{n}.$$

But

$$\cos\frac{(n+1)\,\alpha}{2}=\cos\frac{(n-1)\,\alpha}{2}\cos\alpha-\sin\frac{(n-1)\,\alpha}{2}\sin\alpha.$$

The inequality in question now reduces to $1-\cos^{n-1}\alpha<(\tan((n-1)\alpha/2))\times \tan\alpha$.

Operating with the left term, we have

$$\begin{aligned}1-\cos^{n-1}\alpha<(1-\cos\alpha)(n-1)&=2\left(\sin^2\frac{\alpha}{2}\right)(n-1)\\ &=\left(2\sin\frac{\alpha}{2}\right)\frac{\sin(\alpha/2)}{(\alpha/2)}\left(\frac{(n-1)\,\alpha}{2}\right)\\ &<\left(2\sin\frac{\alpha}{2}\right)\tan\left(\frac{(n-1)\,\alpha}{2}\right).\end{aligned}$$

This will be less than the right term provided

$$2\sin\frac{\alpha}{2}<\frac{(2\sin(\alpha/2))(\cos(\alpha/2))}{\cos\alpha},$$

or $1 < \cos(\alpha/2)/\cos\alpha$. This is certainly true, since

$$\cos\alpha > 0 \qquad \text{for} \quad 0 < \alpha < \pi/n,\ n \geqslant 2.$$

The result follows.

LEMMA 7. *Let the equi-angular path P_4 of m nodes have common angle α. Then φ_{P_4} monotonically increases to its supremum m as $\alpha \downarrow 0$.*

Proof. Let

$$G(\alpha) = \frac{\cos^m\alpha - \cos m\alpha}{\cos^{-m}\alpha - \cos^m\alpha}.$$

Then

$$\varphi_{P_4} = 1 + 2G(\alpha).$$

An application of L'Hospital's rule shows that $\lim_{\alpha\downarrow 0}(1 + 2G(\alpha)) = m$. The lemma will follow if we show that $G(\alpha)$ is strictly decreasing on $0 < \alpha < \pi/m$, if $m > 1$. In the obvious way, we express $G'(\alpha)$ as a quotient whose numerator simplifies to

$$(\cos^{2m}\alpha)(\sin\alpha(1-m)) + \sin\alpha(1+m) - 2\cos^m\alpha\sin\alpha,$$

and whose denominator is greater than 0. So, we wish to show that this expression is less than 0 on $(0, \pi/m)$. Regard it as a quadratic in $\cos^m\alpha$. Proving that it is less than 0 on $0 < \alpha < \pi/m$ is equivalent to showing that $\cos^m\alpha$ is strictly larger than the larger root of

$$\sin\alpha(1-m)\,x^2 - 2(\sin\alpha)\,x + \sin\alpha(1+m) = 0.$$

For this, it is necessary and sufficient that

$$\cos^m\alpha > \frac{\sin m\alpha - \sin\alpha}{\sin(m-1)\,\alpha} \qquad \text{for} \quad 0 < \alpha < \frac{\pi}{m}.$$

This is true by Lemma 6. The result follows.

LEMMA 8. *Any path P that visits a given line more than once has $\varphi_P < N$.*

Proof. Note the very first recurrence of a given projection Q_k in S_m. Then for some $i < j \leqslant m$ one has the subpath

$$(e_1, S_1e_1, ..., Q_kS_ie_1, ..., Q_kS_je_1). \tag{10}$$

If there are $i + 2 \leqslant N$ nodes in the subpath $(e_1, S_1e_1, ..., Q_kS_ie_1)$, there must be at most $(N - (i+2))$ remaining nodes in the path (10).

If l_k is collinear with e_1 and $Q_k S_i e_1$ is on the *ray* determined by e_1, we note that the path $P' = (e_1, S_{i+2}e_1, ..., Q_k S_j e_1, ..., S_m e_1)$ will have a highler ending norm b on the ray r_m determined by $S_m e_1$ than the original path P. Otherwise, Lemmata 2, 3, and 4 show that there exist $i+1$ coplanar, equi-angular rays in the plane generated by e_1, $Q_k S_j e_1$, and the origin, whose resulting (backup free) projection path P^* has a higher ending norm b on r_m. Similarly, there exist $N-i-3$ coplanar, equi-angular (with a possibly different angle β) rays between the end of P^* (or e_1, if the first possibility occurs) and the ray determined by $Q_k S_j e_1$, (with resulting backup free projection path P^{**}), so that the resulting planar projection path P'' formed by splicing P^{**} to P^* has a final ending norm $b \geqslant |S_m e_1|$ on r_m.

We assume for now that P^* and P^{**} do not overlap.

We next shrink β of P^{**} to a value $\beta' < \beta$, so that the resulting path ends on the ray determined by $-e_1$ and is contained entirely in the half plane containing P^*. Splicing this new subpath onto P^* yields a new complete path P''' (which is of type P_2) having both b and l larger than the corresponding values for P'', so $\varphi_{P''} < \varphi_{P'''}$.

A final application of Lemma 4 to the complete path P''' allows us to conclude that there exists an equi-angular path, which we denote by P, of no more than $N+1$ nodes, with $\theta = \pi$ and $\varphi_{P'''} < \varphi_P$.

We have

$$\varphi_P = \frac{1+b}{1-b} = \frac{1+\cos^{N+1}(\pi/(N+1))}{1-\cos^{N+1}(\pi/(N+1))}.$$

Suppose now that P^* and P^{**} overlap. Necessarily, the combined path P'' is contained in a half plane (determined by e_1 and l_k.) We observe that the ending norm b is no greater than that in the above expression, and the ending value for l is no greater than $1+b$. Hence the above expression serves as an upper bound for the value of φ in that case.

The lemma will be proved if we show that the above expression is less than N. This is equivalent to showing that

$$\cos^N \frac{\pi}{N} < 1 - \frac{2}{N} \qquad \text{for} \quad N \geqslant 3.$$

This follows from Lemma 5.

The proof of the Theorem now follows easily from Lemma 7, as we may assume our path is of type P_4 with no more than N nodes.

We conclude this paper with a result which is related to Lemma 2.

Let S_m be a random product from $\mathbf{S} = \mathbf{S}(Q_1, Q_2, ..., Q_N)$ and let P be the corresponding projection path of $m+1$ nodes, *starting at* e_1. The construction in Lemma 2 provides a (unique) planar path P_2 of $m+1$ nodes, starting at e_1 and ending on the ray determined by $S_m e_1$, such that $\varphi_P \leqslant \varphi_{P_2}$. In the case that $N \leqslant 4$, simple examples show that φ_{P_2} may exceed N. We conjecture that for $N \geqslant 4$, the bound of N suffices. At any rate, the following proposition shows that $\varphi_{P_2} \leqslant 1.7N$, for all $N \geqslant 1$.

PROPOSITION 2. *Let* $\mathbf{S}$ *be the algebraic semigroup with infinitely many generators of the form* $U_\alpha Q_j U_\alpha^{-1}$, $1 \leqslant j \leqslant N$, *where* U_α *is a unitary map fixing* e_1. *Then for any word* $W = S_m$ *of* $\mathbf{S}$, *the corresponding projection path* P *which starts at* e_1 *has* $\varphi_P \leqslant 1.7N$.

Proof. For fixed j, each map of the form $U_\alpha Q_j U_\alpha^{-1}$ is a projection onto a line $l_j^{(\alpha)}$ through the origin. Since U_α fixes e_1, the angle between $l_j^{(\alpha)}$ and e_1 is the same as the angle between l_j and e_1. When the map $U_\alpha Q_j U_\alpha^{-1}$ is applied in the product S_m, the corresponding node determines a ray, which, when planarized by the construction of Lemma 2, will coincide with one of the two planarized rays of l_j. Hence despite the new unitary conjugations, the planarized path will have $m+1$ nodes on at most N rays in the first quadrant and at most N rays in the second quadrant. By Lemma 3, we may assume that it is of type P_3. There is no loss of generality in assuming that all $2N+1$ nodes are present and by Lemma 4, we may assume it is of type P_4 ending on a ray whose angle with e_1 is less than or equal to π. Finally, Lemma 7 shows that the first quadrant path P^* of type P_4, of $2N+1$ nodes and ending on the y-axis, has $\varphi_P \leqslant \varphi_{P^*}$. Now

$$\varphi_{P^*} = \frac{l^2}{1-b^2} = \frac{1+\cos^{4N}(\pi/4N)}{1-\cos^{4N}(\pi/4N)}.$$

One can easily check that $\varphi_{P^*} < 1.7N$ for $N = 1$, 2, and 3. In the following it will be assumed that $N \geqslant 4$.

For a constant A, we let $n = 4N$ and note that $\varphi_{P^*} < AN$ if and only if

$$\cos^n \frac{\pi}{n} < 1 - \frac{8}{An+4}. \tag{11}$$

From the proof of Lemma 5, we see that (11) is true if

$$A > \frac{64}{P_n \pi^2(4 - \pi^2/n + \pi^2/n^2)} - \frac{4}{n} = h_1(n),$$

where

$$P_n = \left(\frac{\sin(\pi/2n)}{(\pi/2n)}\right)^2.$$

As P_n is increasing, we see that

$$h_2(n) = \frac{64}{P_{16}\pi^2(4-\pi^2/n)} - \frac{4}{n} > h_1(n), \qquad \text{for} \quad n \geq 16.$$

Hence if the constant A is greater than $h_2(n)$, for $n \geq 16$, then $\varphi_{P*} < AN$, for $N \geq 4$. Easy calculus shows that $h_2(x)$ is decreasing for $x \geq 16$. Numerical computation gives $h_2(16) \approx 1.676$. Hence $\varphi_{P*} < 1.7N$ for $N = (1, 2, \text{and } 3), 4, 5, \ldots$.

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